SEMISPECTRAL MEASURES AS CONVOLUTIONS AND THEIR MOMENT OPERATORS

JUKKA KIUKAS, PEKKA LAHTI, AND KARI YLINEN

Abstract. The moment operators of a semispectral measure having the structure of the convolution of a positive measure and a semispectral measure are studied, with paying attention to the natural domains of these unbounded operators. The results are then applied to conveniently determine the moment operators of the Cartesian margins of the phase space observables.

Keywords: moment operators, convolution, semispectral measure, phase space observables

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1. Introduction

The increasingly accepted view of a quantum observable as a positive operator measure as opposed to the more traditional approach using only spectral measures has added a great deal to our understanding of the mathematical structure and foundational aspects of quantum mechanics. In many cases an observable that is not itself projection valued, nevertheless arises as an unsharp or smeared version of a spectral measure. One way to realize such a smearing is to convolve the spectral measure with a probability measure. In particular, the marginal observables of a phase space observable have such a structure.

Phase space observables have several important applications in quantum mechanics, ranging from the theory of Husimi distributions in quantum optics to state tomography, phase space quantizations, and to the theory of approximate joint measurements of position and momentum, as highlighted, for instance, by the monographs [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. Optical implementations of such observables are also well understood, as described e.g. in a recent study [12], and their mathematical structure has been investigated with great detail [13, 14, 15, 16].

The moments of measurement statistics of an observable are related to the moment operators of that observable in the same way as the outcome probabilities are related to the observable itself. In some cases, the moments may even carry the entire information on the observable [17, 18]. Hence, it makes sense to study the moment operators of a semispectral measure obtained as a convolution, which is the aim of this paper. In Sect. 2, we give the technical lemmas needed for the main results. In particular, we discuss the difficulties in the integrability questions associated with convolutions of nonpositive scalar measures. In Sect. 3, we consider the case of a general convolved semispectral measure by using the operator integral of [19], and in Sect. 4, we work out the Cartesian marginal moment operators for a class of phase space observables.

2. Preliminaries

To begin with, we recall the notion of the convolution of scalar measures, and we prove a lemma on their moment integrals.

The *convolution* of two complex Borel measures $\mu, \nu : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ is the measure $\mu * \nu : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$, defined by

$$\mu * \nu(X) = (\mu \times \nu)(\{(x, y) \mid x + y \in X\}), \quad X \in \mathcal{B}(\mathbb{R}),$$

where $\mu \times \nu$ is the product measure defined on $\mathcal{B}(\mathbb{R}^2)$, the Borel σ-algebra of $\mathbb{R}^2$ (see e.g. [20, p. 648, Definition 8]).

Lemma 1. *Let $\mu, \nu : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ be two complex measures, and let $k \in \mathbb{N}$.*

(a) *A Borel function $f : \mathbb{R} \to \mathbb{C}$ is $\mu * \nu$-integrable if $(x, y) \mapsto f(x + y)$ is integrable with respect to the product measure $\mu \times \nu : \mathcal{B}(\mathbb{R}^2) \to \mathbb{C}$. In that case,*

$$\int f(x)\, d(\mu * \nu)(x) = \int f(x + y)\, d(\mu \times \nu)(x, y).$$

(b) *The function $(x, y) \mapsto (x + y)^k$ is $\mu \times \nu$-integrable, if and only if $x \mapsto x^k$ is both μ- and ν-integrable. In that case,*

$$\int (x + y)^k d(\mu \times \nu)(x, y) = \sum_{n=0}^{k} \binom{k}{n} \left(\int x^{k-n} d\mu(x) \right) \left(\int y^n d\nu(y) \right).$$

Proof. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ denote addition, i.e. $\phi(x, y) = x + y$. Write the product measure $\mu \times \nu$ in terms of the positive and negative parts of its real and imaginary parts:

$$\mu \times \nu = \nu_1 + i\nu_2 = \nu_1^+ - \nu_1^- + i(\nu_2^+ - \nu_2^-),$$

where $\nu_i^\pm = \frac{1}{2}(|\nu_i| \pm \nu_i)$. Then

$$\mu * \nu(X) = \nu_1^+(\phi^{-1}(X)) - \nu_1^-(\phi^{-1}(X)) + i(\nu_2^+(\phi^{-1}(X)) - \nu_2^-(\phi^{-1}(X))), \quad X \in \mathcal{B}(\mathbb{R}). \tag{1}$$

Assume now that $f \circ \phi$ is $\mu \times \nu$-integrable. Then $f \circ \phi$ is integrable with respect to each $\nu_i^\pm$, and

$$\int f \circ \phi\, d(\mu \times \nu) = \int f \circ \phi\, d\nu_1^+ - \int f \circ \phi\, d\nu_1^- + i(\int f \circ \phi\, d\nu_2^+ - \int f \circ \phi\, d\nu_2^-). \tag{2}$$

Since the measures $\nu_i^\pm$ are positive, it follows that f is integrable with respect to each induced measure $X \mapsto \nu_1^+(\phi^{-1}(X))$, and the corresponding integrals are equal (see e.g. [21, p. 163]). Now (1) and (2) imply that f is $\mu * \nu$-integrable, with

$$\int f d(\mu * \nu) = \int f \circ \phi\, d(\mu \times \nu).$$

This proves (a).

To prove (b), suppose first that $(x, y) \mapsto \phi(x, y)^k$ is $\mu \times \nu$-integrable. Since both measures μ and ν are finite, it follows from [20, p. 193, Theorem 13] that $x \mapsto \phi(x, y)^k = (x + y)^k$ is μ-integrable for ν-almost all $y \in \mathbb{R}$. Take any such $y \in \mathbb{R}$. Now $x \mapsto |x + y|^k$ is also $|\mu|$-integrable, where $|\mu|$ denotes the total variation measure of μ. There are positive constants M and K satisfying

$$|x^k| \leq K|x + y|^k + M, \quad x \in \mathbb{R}.$$

This implies that $x \mapsto |x|^k$ is $|\mu|$-integrable, and hence also μ-integrable. It is similarly seen that $x \mapsto |x|^k$ is ν-integrable.

Suppose now that $x \mapsto x^k$ is both μ- and ν-integrable. Since $x \mapsto |x|^k$ is now $|\mu|$- and $|\nu|$-integrable, and these are finite positive measures, it follows that $x \mapsto |x|^l$ is $|\mu|$- and $|\nu|$-integrable for all $l \in \mathbb{N}$, $l \leq k$. Hence, $(x, y) \mapsto |x^l y^m|$ is $|\mu| \times |\nu|$-integrable for all $l, m \in \mathbb{N}$, $l \leq k$,

$m \leq k$. Since $|x+y|^k \leq \sum_{n=0}^{k} \binom{k}{n} |x^{k-n} y^n|$, this implies that $(x, y) \mapsto |\phi(x, y)^k| = |(x+y)^k|$ is $|\mu| \times |\nu|$-integrable. But $|\mu| \times |\nu| = |\mu \times \nu|$ by [20, p. 192, Lemma 11], so $x \mapsto \phi(x, y)^k$ is $\mu \times \nu$-integrable.

The claimed formula follows now easily, since we have shown above that the equivalent integrability conditions imply that $(x, y) \mapsto x^l y^m$ is $\mu \times \nu$-integrable for all $l, m \in \mathbb{N}$, $l \leq k$, $m \leq k$.

□

The converse implication in part (a) of the above lemma does not hold if the measures μ and ν are not assumed to be positive. This is the conclusion of the brief discussion we now enter. Denote $\phi(x+y) = x+y$ as before, and $\Sigma = \{\phi^{-1}(X) \mid X \in \mathcal{B}(\mathbb{R})\}$. Then Σ is a σ-algebra (properly) contained in $\mathcal{B}(\mathbb{R}^2)$. Let μ and ν be complex Borel measures on $\mathbb{R}$ and λ_2 their convolution. Lemma 8 in [20, p. 182] states that the formula $\lambda_1(\phi^{-1}(X)) = \lambda_2(X)$ gives a well-defined complex measure on Σ. (To be precise, the lemma requires the additional assumption that ϕ be surjective, but of course this holds in our situation.) The same lemma says that the total variations satisfy $|\lambda_1|(\phi^{-1}(X)) = |\lambda_2|(X)$ for all $X \in \mathcal{B}(\mathbb{R})$, and moreover for any λ_2-integrable Borel function $f : \mathbb{R} \to \mathbb{C}$, the composite function $f \circ \phi$ is λ_1-integrable, and the natural integral transformation formula holds.

Obviously λ_1 is just the restriction of the product measure $\mu \times \nu$ to Σ. Any Σ-measurable function $g : \mathbb{R}^2 \to \mathbb{C}$ which is $\mu \times \nu$-integrable, is integrable with respect to the restriction of the variation measure $|\mu \times \nu|$ to Σ. In the following example we see that this need not be the case if g is just assumed to be λ_1-integrable. This phenomenon is at the root of the fact that the implication in Lemma 1 (a) cannot be reversed.

Example 1. We construct two discrete measures μ and ν supported by $\mathbb{Z}$. Let $\sum_{k=0}^{\infty} a_k$ be any convergent series with positive terms, and define $b_{2k} = b_{2k+1} = a_k$ for all $k = 0, 1, 2, \ldots$, and $b_k = 0$ if $k \in \mathbb{Z}$, $k \leq -1$. We set $\mu(\{n\}) = b_n$ for all $n \in \mathbb{Z}$. The discrete measure ν is defined by setting $\nu(\{n\}) = (-1)^n b_{-n}$ for all $n \in \mathbb{Z}$. Then the convolution $\lambda = \mu * \nu$ is supported by $\mathbb{Z}$, and we have $\lambda(\{n\}) = \sum_{j=-\infty}^{\infty} b_j (-1)^{n-j} b_{j-n} = \sum_{j=0}^{\infty} b_j (-1)^{n-j} b_{j-n}$. If n is even, it follows that $\lambda(\{n\}) = 0$, since $b_{2k} = b_{2k+1}$. However, $c_n = \sum_{j=-\infty}^{\infty} |b_j (-1)^{n-j} b_{j-n}| > 0$. We now define $f : \mathbb{R} \to \mathbb{C}$ by setting $f(2k) = c_{2k}^{-1}$ for all $k \in Z$ and $f(x) = 0$ if $x \in \mathbb{R} \setminus 2\mathbb{Z}$. Then $\int_{\mathbb{R}} f(x) d\lambda(x) = 0$, but the function $(x, y) \mapsto f(x+y)$ is not $|\mu \times \nu|$-integrable, since its integral with respect to $|\mu \times \nu|$ over any set $\{(x, y) \mid x+y = n\}$, $n \in 2\mathbb{Z}$, equals 1.

To close this preliminary section, we recall the notion of an operator integral in the sense of [19]. Let Ω be a set and $\mathcal{A}$ a σ-algebra of subsets of Ω. Let $E : \mathcal{A} \to L(\mathcal{H})$ be a semispectral measure (normalized positive operator measure) taking values in $L(\mathcal{H})$, the set of bounded operators on a complex Hilbert space $\mathcal{H}$ ($\neq \{0\}$). Thus, for any $\varphi, \psi \in \mathcal{H}$, the set function $X \mapsto E_{\psi,\varphi}(X) := \langle \psi \mid E(X)\varphi \rangle$ is a complex measure. For any measurable function $f : \Omega \to \mathbb{C}$ we let $D(f, E)$ denote the set of those vectors $\varphi \in \mathcal{H}$ for which f is $E_{\psi,\varphi}$-integrable for all $\psi \in \mathcal{H}$. The set $D(f, E)$ is a vector subspace of $\mathcal{H}$ and the formula

$$\langle \psi \mid L(f, E)\varphi \rangle = \int_{\Omega} f \, dE_{\psi,\varphi}, \quad \varphi \in D(f, E), \psi \in \mathcal{H},$$

defines a unique linear operator $L(f, E)$, with the domain $D(f, E)$. The set $\tilde{D}(f, E) = \{\varphi \in \mathcal{H} \mid \int |f|^2 \, dE_{\varphi,\varphi} < \infty\}$ is a subspace of $D(f, E)$, and we let $\tilde{L}(f, E)$ denote the restriction of $L(f, E)$ into $\tilde{D}(f, E)$. We recall that if E is a spectral (projection valued) measure, then

$\tilde{D}(f,E) = D(f,E)$ and the operator $L(f,E)$ is densely defined. We consider here only the cases where $(\Omega, \mathcal{A})$ is $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ or $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

3. Convolutions and their moment operators

For any $X \in \mathcal{B}(\mathbb{R})$, let χ_X denote the characteristic function of X. Recall that ϕ denotes the map $(x,y) \mapsto x+y$. The function $\chi_X \circ \phi$ is bounded and thereby integrable with respect to the product measure. Hence Lemma 1(a) and Fubini's theorem give that the function

$$y \mapsto \mu(X-y) = \int \chi_{X-y}(x)\, d\mu(x) = \int \chi_X(x+y)\, d\mu(x)$$

coincides almost everywhere with a Borel function, and

$$\mu * \nu(X) = \int \left(\int \chi_X(x+y)\, d\mu(x) \right) d\nu(y) = \int_{\mathbb{R}} \mu(X-y) d\nu(y), \quad X \in \mathcal{B}(\mathbb{R}).$$

Let now $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a semispectral measure, and let $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ be a probability measure. Since the sesquilinear form

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \mu(X-y)\, dE_{\psi,\varphi}(y)$$

is clearly bounded, one can define $\mu * E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ via $\langle \varphi | (\mu * E)(X) \psi \rangle := \mu * E_{\psi,\varphi}(X)$, $\varphi, \psi \in \mathcal{H}$. It follows from the monotone convergence theorem that $\mu * E$ is a semispectral measure.

Denote $\mu[k] := \int x^k\, d\mu(x)$, in case this integral exists (i.e. when $\int |x^k|\, d\mu(x) < \infty$).

Proposition 1. *Let $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a semispectral measure, and $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ a probability measure. Then*

(a) *$\tilde{D}(x^k, \mu * E)$ equals either $\tilde{D}(x^k, E)$ or $\{0\}$, depending on whether $\mu[2k]$ exists or not. In the former case,*

$$\tilde{L}(x^k, \mu * E) = \sum_{n=0}^{k} \binom{k}{n} \mu[k-n] \tilde{L}(x^n, E).$$

(b) *If $\mu[k]$ exists, then $D(x^k, E) \subset D(x^k, \mu * E)$, and*

$$L(x^k, \mu * E) \supset \sum_{n=0}^{k} \binom{k}{n} \mu[k-n] L(x^n, E).$$

Proof. Since $(\mu * E)_{\varphi,\varphi} = \mu * E_{\varphi,\varphi}$ by definition, and these measures are positive, it follows from e.g. [21, p. 163] that x^{2k} is $(\mu * E)_{\varphi,\varphi}$-integrable if and only if $(x,y) \mapsto (x+y)^{2k}$ is $\mu \times E_{\varphi,\varphi}$-integrable. By Lemma 1 (b), this happens exactly when $\mu[2k]$ exists and $\varphi \in \tilde{D}(x^k, E)$. Hence, $\tilde{D}(x^k, \mu * E)$ equals either $\tilde{D}(x^k, E)$ or $\{0\}$, depending on whether $\mu[2k]$ exists or not. Suppose now that $\mu[2k]$ exists, and let $\varphi \in \tilde{D}(x^k, \mu * E) = \tilde{D}(x^k, E)$. Since this set is contained in $D(x^k, E)$, it follows that x^k is $E_{\psi,\varphi}$-integrable for all $\psi \in \mathcal{H}$. Also, $\mu[k]$ clearly exists. Hence, according to Lemma 1 (b), $(x,y) \mapsto (x+y)^k$ is $\mu \times E_{\psi,\varphi}$-integrable for all $\psi \in \mathcal{H}$, so using both (a) and (b) of that lemma, we get

$$(3) \quad \int x^k\, d(\mu * E)_{\psi,\varphi} = \int (x+y)^k\, d(\mu \times E_{\psi,\varphi})(x,y) = \sum_{n=0}^{k} \binom{k}{n} \mu[k-n] \int x^n\, dE_{\psi,\varphi}, \quad \psi \in \mathcal{H}.$$

This completes the proof of (a). To prove (b), suppose that $\mu[k]$ exists, so that x^k is μ-integrable. Now if $\varphi \in D(x^k, E)$, then x^k is also $E_{\psi,\varphi}$-integrable for any $\psi \in \mathcal{H}$. According to Lemma 1 (b), this implies that $(x,y) \mapsto (x+y)^k$ is $\mu \times E_{\psi,\varphi}$-integrable for all $\psi \in \mathcal{H}$, and using again also Lemma 1 (a), we see that x^k (and thus also x^n with $n \leq k$) is $\mu * E_{\psi,\varphi}$-integrable (i.e. $(\mu * E)_{\psi,\varphi}$-integrable) for all $\psi \in \mathcal{H}$, and the relation (3) holds. But this means that we have proved (b). $\square$

Proposition 2. *Let $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ be a spectral measure, let $k \in \mathbb{N}$, and let $\mu : \mathcal{B}(\mathbb{R}) \to [0,1]$ be a probability measure such that $\mu[k]$ exists. Denote $A = L(x, E)$. Then*

$$L(x^k, \mu * E) = \sum_{n=0}^{k} \binom{k}{n} \mu[k-n] A^n, \quad D(x^k, \mu * E) = D(A^k).$$

Moreover, $\tilde{D}(x^k, \mu * E)$ equals either $D(A^k) = D(x^k, \mu * E)$ or $\{0\}$, depending on whether $\mu[2k]$ exists or not.

Proof. Since E is a spectral measure, A is selfadjoint, and $D(A^k) = D(x^k, E) = \tilde{D}(x^k, E)$, $L(x^k, E) = A^k$ for all $k \in \mathbb{N}$. According to the preceding proposition (b), $L(x^k, \mu * E)$ is a symmetric extension of the selfadjoint operator $\sum_{n=0}^{k} \binom{k}{n} \mu[k-n] A^n$. Thus these operators must be equal. The last claim follows immediately from part (a) of the preceding proposition. $\square$

Remark 1. Let $E : B(\mathbb{R}) \to L(\mathcal{H})$ be any spectral measure, and choose a probability measure μ such that $\mu[k]$ exists but $\mu[2k]$ does not. Then $L(x^k, \mu * E)$ is a densely defined selfadjoint operator, but $\tilde{D}(x^k, \mu * E) = \{0\}$.

Consider then the following special case. For any positive operator T of trace one, and a selfadjoint operator A in $\mathcal{H}$, let $p_T^A : \mathcal{B}(\mathbb{R}) \to [0,1]$ be the probability measure defined by $p_T^A(X) = \mathrm{Tr}[T E^A(X)]$, where E^A is the spectral measure of A.

Let A be a selfadjoint operator and $k \in \mathbb{N}$, such that $p_T^A[k]$ exists. According to e.g. [22, Lemma 1] and [23, Lemma 1], this happens exactly when $\sqrt{|A|}^k \sqrt{T}$ is a Hilbert-Schmidt operator. Under this condition, we then have, according to the preceding proposition, that

$$L(x^k, p_T^A * E^B) = \sum_{n=0}^{k} \binom{k}{n} p_T^A[k-n] B^n, \quad D(x^k, p_T^A * E^B) = D(B^k)$$

for any selfadjoint operator B. Moreover, $\tilde{D}(x^k, p_T^A * E^B) \neq \{0\}$ if and only if $p_T^A[2k]$ exists, or, equivalently, $A^k \sqrt{T}$ is a Hilbert-Schmidt operator. This stronger condition assures also that $p_T^A[k-n] = \mathrm{Tr}[A^{k-n} T]$ in the above formula, the operators $A^{k-n} T$ being in the trace class.

Remark 2. As an example, take $T = |\eta\rangle\langle\eta|$ with $\eta \in D(\sqrt{|A|})$ but $\eta \notin D(A)$. Then $L(x, p^A_{|\eta\rangle\langle\eta|} * E^B) = B$, since $\sqrt{|A|}\sqrt{|\eta\rangle\langle\eta|} = \sqrt{|A|}|\eta\rangle\langle\eta|$ is clearly a Hilbert-Schmidt operator. However, the square integrability domain is $\{0\}$, since $|A|\sqrt{|\eta\rangle\langle\eta|}$ is quite far from being Hilbert-Schmidt (its domain is $\{0\}$). Note also that now $p^A_{|\eta\rangle\langle\eta|}[1]$ is not equal to $\mathrm{Tr}[A|\eta\rangle\langle\eta|]$, since this trace is not even defined.

4. Phase space observables

Let $\mathcal{H} = L^2(\mathbb{R})$, and let Q and P be the selfadjoint position and momentum operators in $\mathcal{H}$, and $W(q,p)$, $(q,p) \in \mathbb{R}^2$, the corresponding Weyl operators. Consider now the phase space

observable $E^T:\mathcal{B}(\mathbb{R}^2)\to L(\mathcal{H})$,

$$E^T(Z)=\frac{1}{2\pi}\int_Z W(q,p)TW(q,p)^*\,dqdp,$$

with T a positive operator of trace one. The Cartesian marginal measures $E^{T,x},E^{T,y}:\mathcal{B}(\mathbb{R})\to L(\mathcal{H})$ are defined by $E^{T,x}(X):=E^T(X\times\mathbb{R})$, $E^{T,y}(Y):=E^T(\mathbb{R}\times Y)$. It is well known that they are equal to $p_T^{-Q}*E^Q$ and $p_T^{-P}*E^P$, respectively, see e.g. [1, Theorem 3.4.2]. According to the above discussion, we can thus determine the kth moment operators of the x- and y- margins, under the respective conditions that $p_T^{-Q}[k]$ and $p_T^{-P}[k]$ exist, or, equivalently, $\sqrt{|Q|}^k\sqrt{T}$ and $\sqrt{|P|}^k\sqrt{T}$ are Hilbert-Schmidt:

Proposition 3. *Let $k\in\mathbb{N}$.*

(a) *If $\sqrt{|Q|}^k\sqrt{T}$ is a Hilbert-Schmidt operator, then*

$$L(x^k,E^{T,x})=\sum_{n=0}^{k}\binom{k}{n}(-1)^{k-n}p_T^Q[k-n]Q^n,\quad D(x^k,E^{T,x})=D(Q^k).$$

(b) *Part (a) holds also when "x" and "Q" are replaced by "y" and "P".*

Under the square integrability condition that $Q^k\sqrt{T}$ (respectively $P^k\sqrt{T}$) be Hilbert-Schmidt, we get $p_T^Q[k-n]=\mathrm{Tr}[Q^{k-n}T]$ ($p_T^P[k-n]=\mathrm{Tr}[P^{k-n}T]$).

Remark 3. According to the discussion in the preceding remark, a simple example where $L(x,E^{T,x})=Q$ but $\tilde{D}(x,E^{T,x})=\{0\}$, is obtained by taking $T=|\eta\rangle\langle\eta|$, where $\eta\in\mathcal{H}$ is a unit vector with $\int|x||\eta(x)|^2\,dx<\infty$, $\int x|\eta(x)|^2\,dx=0$, and $\int x^2|\eta(x)|^2\,dx=\infty$.

An additional problem with the moment operators $L(x^k,E^{T,x})$ and $L(x^k,E^{T,y})$ is their connection to the operators $L(x^k,E^T)$ and $L(y^k,E^T)$, which we have considered before (see [22, 23]). By writing e.g. $E^{T,x}(X)=E^T(\pi_1^{-1}(X))$ where $\pi_1:\mathbb{R}^2\to\mathbb{R}$ is the coordinate projection $(x,y)\mapsto x$, we notice that a similar "change of variables" argument as that in Lemma 1 gives $L(x^k,E^{T,x})\supset L(x^k,E^T)$. Now if $\sqrt{|Q|}^k\sqrt{T}$ is Hilbert-Schmidt, then we know from the above proposition that $L(x^k,E^{T,x})$ is a selfadjoint operator, a polynomial in Q. However, this does not determine $L(x^k,E^T)$; we can only say that it has $L(x^k,E^{T,x})$ as a selfadjoint extension.

Consider then the square integrability domains. Since the measures involved are now positive, the "change of variables formula" (see e.g. [21, p. 163]) can be used to conclude that the restrictions are equal: $\tilde{L}(x^k,E^{T,x})=\tilde{L}(x^k,E^T)$. According to Proposition 2, this operator is nontrivial exactly when $|Q|^k\sqrt{T}$ is Hilbert-Schmidt, in which case it is selfadjoint. This stronger condition then forces both the symmetric extensions $L(x^k,E^{T,x})$ and $L(x^k,E^T)$ to coincide with the restriction, and we recover Theorem 4 of [23].

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Jukka Kiukas, Department of Physics and Astronomy, University of Turku, FI-20014 Turku, Finland. At present: Institut für Mathematische Physik, TU Braunschweig, DE-38106 Braunschweig, Germany.
E-mail address: jukka.kiukas@utu.fi

Pekka Lahti, Department of Physics and Astronomy, University of Turku, FI-20014 Turku, Finland
E-mail address: pekka.lahti@utu.fi

Kari Ylinen, Department of Mathematics, University of Turku, FI-20014 Turku, Finland
E-mail address: kari.ylinen@utu.fi